

Multivalued Elliptic Equation with exponential critical growth in \mathbb{R}^2 *

Claudianor O. Alves[†] and Jefferson A. Santos

Abstract

In this work we study the existence of nontrivial solution for the following class of multivalued elliptic problems

$$-\Delta u + V(x)u - \epsilon h(x) \in \partial_t F(x, u) \quad \text{in } \mathbb{R}^2, \quad (P)$$

where $\epsilon > 0$, V is a continuous function verifying some conditions, $h \in (H^1(\mathbb{R}^2))^*$ and $\partial_t F(x, u)$ is a generalized gradient of $F(x, t)$ with respect to t and $F(x, t) = \int_0^t f(x, s) ds$. Assuming that f has an exponential critical growth and a discontinuity point, we have applied Variational Methods for locally Lipschitz functional to get two solutions for (P) when ϵ is small enough.

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1 Introduction

The interest in the study of nonlinear partial differential equations with discontinuous nonlinearities has increased because many free boundary problems arising in mathematical physics may be stated in this form. Among these problems, we have the obstacle problem, the seepage surface problem, and the Elenbaas equation; see for example [15, 16, 17].

Among the typical examples, we have chosen the model for the heat conductivity in electrical media. This model has a discontinuity in its constitutive laws. In fact, considering a domain $\Omega \subset \mathbb{R}^2$ (which in particular could be taken as being the whole space \mathbb{R}^2 , see [10]) with electrical media, the thermal and electrical conductivity are denoted by $K(x, t)$ and $\sigma(x, t)$, respectively. Here x is in Ω and t represents the temperature. Since we are considering an electrical media, the function σ may have discontinuities in t ,

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and the distribution of the temperature is unknown. The differential equation describing this distribution is

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i} \left((K(x, u(x))) \frac{\partial u(x)}{\partial x_i} \right) = \sigma(x, u(x)).$$

Note that this equation is related to a free boundary problem in which the jump surface of the electrical conductivity is unknown. We describe this surface as being the set

$$\Gamma_\alpha(u) = \{x \in \Omega : u(x) = \alpha, \quad \sigma \text{ is discontinuous at } \alpha\}. \quad (1.1)$$

When the thermal conductivity K is constant, $\Omega = \mathbb{R}^2$ and the electrical conductivity σ is given by

$$\sigma(x, t) = H(t - a)f(t) + \epsilon h(x) - V(x)t,$$

with f having an exponential critical growth, the model becomes

$$-\Delta u + V(x)u = H(u - a)f(u) + \epsilon h(x) \quad \text{in } \mathbb{R}^2. \quad (1.2)$$

Here H is the Heaviside function, that is,

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

ϵ is a positive parameter and h is a measurable function defined in \mathbb{R}^2 . Note that in this model the jump surface of the solution (1.1) is represented by the set

$$\Gamma_a(u) = \{x \in \mathbb{R}^2 : u(x) = a\}. \quad (1.3)$$

Related to Problem (1.2) for the special case of $a = 0$, i.e., without jump discontinuities, we cite the works by de Freitas [24] and do Ó, de Medeiros and Severo [28, 29].

A rich literature is available by now on problems with discontinuous nonlinearities, and we refer the reader to Alves, Bertone and Gonçalves [4], Alves and Bertone [5], Alves, Gonçalves and Santos [6], , Ambrosetti and Turner [9], Ambrosetti, Calahorrano and Dobarro [10], Badiale and Tarantelo [11], Carl, Le and Motreanu [13], Clarke [14], Chang [15], Carl and Dietrich [18], Carl and S. Heikkilä [19, 20], Cerami [21], de Souza, de Medeiros and Severo [25, 26], Hu, Kourougenis and Papageorgiou [33], Motreanu and Vargas [35], Radulescu [37] and their references. Several techniques have been developed or applied in their study, such as variational methods for nondifferentiable functionals, lower and upper solutions, global branching, fixed point theorem, and the theory of multivalued mappings.

After a bibliography review, we did not find any paper involving existence of solution for a class of elliptic problem with discontinuous nonlinearity and exponential critical growth via variational methods for nondifferentiable

functional. Motivated by this fact, in this paper we employ variational techniques to study existence and multiplicity of nonnegative solutions for a large class of multivalued elliptic equations, which includes the equation (1.2). More precisely, we will study the multivalued elliptic equation

$$-\Delta u + V(x)u - \epsilon h(x) \in \partial_t F(x, u) \quad \text{in } \mathbb{R}^2, \quad (P)$$

where $\epsilon > 0$ is a positive parameter, V is a continuous function verifying some technical conditions, $h \in (H^1(\mathbb{R}^2))^*$ and $F(x, t)$ is the primitive of a function $f(x, t)$, which has an exponential critical growth and a discontinuity point, for more details see Section 2.

In \mathbb{R}^2 , to apply variational methods, the natural growth restriction on the function f is given by the inequality of Trudinger and Moser [34, 38]. More precisely, we say that a function f has an exponential critical growth if there is $\alpha_0 > 0$ such that

$$\lim_{|s| \rightarrow \infty} \frac{|f(s)|}{e^{\alpha s^2}} = 0 \quad \forall \alpha > \alpha_0 \quad \text{and} \quad \lim_{|s| \rightarrow \infty} \frac{|f(s)|}{e^{\alpha s^2}} = +\infty \quad \forall \alpha < \alpha_0.$$

We would like to mention that problems involving exponential critical growth, with f being a continuous functions, have received a special attention at last years, see for example, [2, 7, 8, 12, 22, 23, 30, 31] and their references. Here, since we intend to get a solution for the differential inclusion (P), we assume that there exists $\alpha_0 > 0$ such that

$$(f_0) \quad \limsup_{t \rightarrow +\infty} \frac{\max \{|\xi|; \xi \in \partial_t F(x, t)\}}{e^{\alpha_0 |t|^2}} < +\infty \quad \text{uniformly in } x \in \mathbb{R}^2.$$

Moreover, assuming a condition at origin like

$$(f_1) \quad \limsup_{t \rightarrow 0} \frac{2 \max \{|\xi|; \xi \in \partial_t F(x, t)\}}{|t|} < +\infty \quad \text{uniformly in } x \in \mathbb{R}^2.$$

it is easy to check that the functional $\Psi : H^1(\mathbb{R}^2) \rightarrow \mathbb{R}$ given by

$$\Psi(u) = \int_{\mathbb{R}^2} F(x, u) dx$$

is well defined, for more details see Section 4. However, to apply variational methods is better to consider the functional Ψ in a more appropriated domain, that is, $\Psi : L^\Phi(\mathbb{R}^2) \rightarrow \mathbb{R}$, for $\Phi(t) = e^{|t|^2} - 1$. But, once Φ does not satisfy the Δ_2 -condition, we cannot guarantee that given $J \in (L^\Phi(\mathbb{R}^2))^*$, then

$$J(u) = \int_{\mathbb{R}^2} v u dx, \quad \forall u \in L^\Phi(\mathbb{R}^2),$$

for some $v : \mathbb{R}^2 \rightarrow \mathbb{R}$ mensurable function. For the familiar readers with the study of the differential inclusions, they will observe that the above remark is

bad to apply variational methods, because in general for this type of equations we need to prove that the inclusion below holds

$$\partial\Psi(u) \subset \partial_t F(x, u) = [\underline{f}(x, u(x)), \overline{f}(x, u(x))] \text{ a.e. in } \mathbb{R}^2,$$

where

$$\underline{f}(x, t) = \lim_{r \downarrow 0} \text{ess inf} \{f(x, s); |s - t| < r\}$$

and

$$\overline{f}(x, t) = \lim_{r \downarrow 0} \text{ess sup} \{f(x, s); |s - t| < r\}.$$

In Section 4, we analyze this question. In fact, we show that it is enough to consider $\Psi : E_\Phi(\mathbb{R}^2) \rightarrow \mathbb{R}$ where

$$E_\Phi(\Omega) = \overline{C_0^\infty(\mathbb{R}^2)}^{\|\cdot\|_\Phi}.$$

Before to state our main result, we must mention our conditions on V, h and f , which are the following:

$$(h_0) \quad h \in (H^1(\mathbb{R}^2))^* \text{ and } 0 < \int_{\mathbb{R}^2} h \, dx < +\infty.$$

$$(V_1) \quad V \text{ is continuous and } V(x) \geq V_0 > 0, \forall x \in \mathbb{R}^2,$$

$$(V_2) \quad \frac{1}{V} \in L^1(\mathbb{R}^2).$$

$$(f_2) \quad \text{There is } t_0 \geq 0 \text{ such that}$$

$$f(x, t) = 0 \quad \text{for } t < t_0 \text{ and } \forall x \in \mathbb{R}^2$$

and

$$f(x, t) > 0 \quad \text{for } t > t_0 \text{ and } \forall x \in \mathbb{R}^2.$$

$$(f_3) \quad \limsup_{t \rightarrow 0} \frac{2 \max \{|\xi|; \xi \in \partial_t F(x, t)\}}{|t|} < \lambda_1 \text{ uniformly in } x \in \mathbb{R}^2, \text{ where}$$

$$\lambda_1 = \inf_{u \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2) \, dx}{\int_{\mathbb{R}^2} |u|^2 \, dx}$$

and

$$E := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)u^2 \, dx < +\infty \right\}.$$

$$(f_4) \quad \text{There is a compact set } K \subset \mathbb{R}^2 \text{ and constants } c_3, c_4 > 0 \text{ and } \nu > 2, \text{ such that}$$

$$F(x, t) \geq c_3 t^\nu - c_4, \quad \text{for } t \geq 0 \text{ and } \forall x \in K.$$

$$(f_5) \quad \text{There is } \tau > 2 \text{ verifying}$$

$$0 \leq \tau F(x, t) \leq \underline{f}(x, t)t, \quad \text{for } t \geq t_0 \text{ and } \forall x \in \mathbb{R}^2.$$

(f₆) There are $p > 2$ and $\mu > 0$ such that

$$F(x, t) \geq \mu(t - t_0)^p, \quad \text{for } t \geq t_0 \text{ and } \forall x \in \mathbb{R}^2.$$

Here, we would like to point out that the function

$$f(t) = 2H(t - a)te^{t^2}, \quad \forall t \in \mathbb{R}, \quad (1.4)$$

verifies (f₁) – (f₆).

Now, we are able to state our main result

Theorem 1.1 *Assume (V₁) – (V₂), (h₀) and (f₀), (f₂) – (f₆). Then, there are ϵ_0, μ^* and $t_1 > 0$, such that problem (P) possesses a solution $v_\epsilon \in E$, with $I_\epsilon(v_\epsilon) = d_\epsilon > 0$, for all $\epsilon \in (0, \epsilon_0)$, $t_0 \in [0, t_1)$ and $\mu \geq \mu^*$. Moreover, decreasing ϵ_0 and t_1 , and increasing μ^* , if necessary, we have two solutions $u_\epsilon, v_\epsilon \in E$ with*

$$I_\epsilon(u_\epsilon) = c_\epsilon < 0 < d_\epsilon = I_\epsilon(v_\epsilon).$$

In the proof of Theorem 1.1, we use variational methods for nondifferentiable functional. A solution is obtained by applying Ekeland's variational principle, while the other one is obtained by using Mountain Pass Theorem. Here, we would like point out that by applying the above theorem for the function f given in (1.4), we find two solutions $u_1, u_2 \in H^1(\mathbb{R}^2)$ for the equation

$$-\Delta u = 2H(u - a)ue^{u^2} + \epsilon h, \quad \text{in } \mathbb{R}^2,$$

with

$$|[u_i = a]| = 0 \quad \text{for } i = 1, 2.$$

Notation: In this paper, we use the following notations:

- The usual norms in $L^t(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$ will be denoted by $|\cdot|_t$ and $\|\cdot\|$ respectively.
- C denotes (possible different) any positive constant.
- $B_R(z)$ denotes the open ball with center at z and radius R .
- If $B \subset \mathbb{R}^2$ is a measurable set, let us denote by $|B|$ the Lebesgue's measure of B .
- Φ denotes the N-function $\Phi(t) = e^{|t|^2} - 1$.

2 Technical results involving the exponential critical growth

In this section, we will prove some technical lemmas, which are crucial in our approach. Since we will work with exponential critical growth, some versions of the Trudinger-Moser inequality are very important in our arguments. The first version that we would like to recall is due to Trudinger and Moser, see [34] and [38], which claims if Ω is a bounded domain with smooth boundary, then for any $u \in H_0^1(\Omega)$,

$$\int_{\Omega} e^{\alpha|u|^2} dx < +\infty, \quad \text{for every } \alpha > 0. \quad (2.1)$$

Moreover, there exists a positive constant $C = C(\alpha, |\Omega|)$ such that

$$\sup_{\|u\|_{H_0^1(\Omega)} \leq 1} \int_{\Omega} e^{\alpha|u|^2} dx \leq C, \quad \forall \alpha \leq 4\pi. \quad (2.2)$$

A version in $H^1(\Omega)$ has been proved by Adimurthi and Yadava [3], and it says that if Ω is a bounded domain with smooth boundary, then for any $u \in H^1(\Omega)$,

$$\int_{\Omega} e^{\alpha|u|^2} dx < +\infty, \quad \text{for every } \alpha > 0. \quad (2.3)$$

Furthermore, there exists a positive constant $C = C(\alpha, |\Omega|)$ such that

$$\sup_{\|u\|_{H^1(\Omega)} \leq 1} \int_{\Omega} e^{\alpha|u|^2} dx \leq C, \quad \forall \alpha \leq 2\pi. \quad (2.4)$$

The third version that we will use is due to Cao [12], which is version of the Trudinger-Moser inequality in whole space \mathbb{R}^2 and has the following statement:

$$\int_{\mathbb{R}^2} (e^{\alpha|u|^2} - 1) dx < +\infty, \quad \text{for all } u \in H^1(\mathbb{R}^2) \text{ and } \alpha > 0. \quad (2.5)$$

Besides, given $\alpha < 4\pi$ and $M > 0$, there is a constant $C_1 = C_1(M, \alpha) > 0$ verifying

$$\sup_{u \in \mathcal{B}_M} \int_{\mathbb{R}^2} (e^{\alpha|u|^2} - 1) dx \leq C_1 \quad (2.6)$$

where

$$\mathcal{B}_M = \{u \in H^1(\mathbb{R}^2) : |\nabla u|_2 \leq 1 \text{ and } |u|_2 \leq M\}.$$

As a consequence from (2.5)-(2.6), we are able to prove some technical lemmas. The first of them is crucial in the study of the (PS) condition for I_{ϵ} .

Lemma 2.1 *Let $\alpha > 0$ and (u_n) be a sequence in $H^1(\mathbb{R}^2)$ with*

$$\limsup_{n \rightarrow +\infty} \|u_n\| < \sqrt{\frac{4\pi}{\alpha}}.$$

Then, there exist $t > 1$, t close to 1, and $C > 0$ satisfying

$$\int_{\mathbb{R}^N} \left(e^{\alpha|u_n|^2} - 1 \right)^t dx \leq C, \quad \forall n \in \mathbb{N}.$$

Proof. As

$$\limsup_{n \rightarrow \infty} \|u_n\| < \sqrt{\frac{4\pi}{\alpha}},$$

there are $m > 0$ and $n_0 \in \mathbb{N}$ verifying

$$\|u_n\|^2 < m < \frac{4\pi}{\alpha}, \quad \forall n \geq n_0.$$

Fix $t > 1$, with t close to 1, and $\beta > t$ satisfying $\beta m < \frac{4\pi}{\alpha}$. Then, there exists $C = C(\beta) > 0$ such that

$$\int_{\mathbb{R}^2} \left(e^{\alpha|u_n|^2} - 1 \right)^t dx \leq C \int_{\mathbb{R}^2} \left(e^{\alpha\beta m \left(\frac{\|u_n\|}{\|u_n\|} \right)^2} - 1 \right) dx,$$

for every $n \geq n_0$. Hence, by (2.6),

$$\int_{\mathbb{R}^2} \left(e^{\alpha|u_n|^2} - 1 \right)^t dx \leq C_1 \quad \forall n \geq n_0,$$

for some positive constant C_1 . Now, the lemma follows fixing

$$C = \max \left\{ C_1, \int_{\mathbb{R}^2} \left(e^{\alpha|u_1|^2} - 1 \right)^t dx, \dots, \int_{\mathbb{R}^2} \left(e^{\alpha|u_{n_0}|^2} - 1 \right)^t dx \right\}.$$

■

Lemma 2.2 Let $\beta, M > 0$ verifying $\beta M < 4\pi$ and $q > 2$. If $\|u\|^2 \leq M$, then there is $C = C(\beta, M, q) > 0$ such that,

$$\int_{\mathbb{R}^2} |u|^q \left(e^{\beta|u|^2} - 1 \right) dx \leq C(\beta) \|u\|^q.$$

Proof. In what follows, fix $t > 1$ close to 1, such that $\alpha = t\beta M < 4\pi$. Then, there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^2} \left(e^{\beta|u|^2} - 1 \right)^t dx \leq C \int_{\mathbb{R}^2} \left(e^{\beta t|u|^2} - 1 \right) dx.$$

Note that

$$\int_{\mathbb{R}^2} \left(e^{\beta t|u|^2} - 1 \right) dx = \int_{\mathbb{R}^2} \left(e^{\beta t \|u\|^2 \left(\frac{u}{\|u\|} \right)^2} - 1 \right) dx \leq \int_{\mathbb{R}^2} \left(e^{\beta t M \left(\frac{u}{\|u\|} \right)^2} - 1 \right) dx.$$

Thereby, by (2.6),

$$\int_{\mathbb{R}^2} \left(e^{\beta t|u|^2} - 1 \right) dx \leq \int_{\mathbb{R}^2} \left(e^{\alpha \left(\frac{u}{\|u\|} \right)^2} - 1 \right) dx \leq \sup_{\|v\| \leq 1} \int_{\mathbb{R}^2} \left(e^{\alpha|v|^2} - 1 \right) dx = C < +\infty.$$

From this, the function $\zeta_u = e^{\beta|u|^2} - 1 \in L^t(\mathbb{R}^2)$ and there is $C = C(\beta, M) > 0$ such that

$$|\zeta_u|_t \leq C, \quad \forall u \in B_M = \{u \in H^1(\mathbb{R}^2) : \|u\| \leq M\}. \quad (2.7)$$

Then, by applying the Hölder's inequality,

$$\int_{\mathbb{R}^2} |u|^q \left(e^{\beta|u|^2} - 1 \right) dx = \int_{\mathbb{R}^2} |u|^q \zeta_u dx \leq |\zeta_u|_t |u|_{t',q}^q$$

where $\frac{1}{t} + \frac{1}{t'} = 1$. Hence, by Sobolev embedding, there is $C > 0$ such that

$$\int_{\mathbb{R}^2} |u|^q \left(e^{\beta|u|^2} - 1 \right) dx \leq C |\zeta_u|_t \|u\|^q. \quad (2.8)$$

Now, the lemma follows combining (2.7) and (2.8). ■

3 Preliminaries about Orlicz spaces

In this section, we recall some properties of Orlicz and Orlicz-Sobolev spaces. We refer to [1, 27, 32, 36] for the fundamental properties of these spaces. First of all, we recall that a continuous function $A : \mathbb{R} \rightarrow [0, +\infty)$ is a N-function if:

- (i) A is convex.
- (ii) $A(t) = 0 \Leftrightarrow t = 0$.
- (iii) $\lim_{t \rightarrow 0} \frac{A(t)}{t} = 0$ and $\lim_{t \rightarrow +\infty} \frac{A(t)}{t} = +\infty$.
- (iv) A is even.

We say that a N-function A verifies the Δ_2 -condition, denote by $A \in \Delta_2$, if

$$A(2t) \leq K_* A(t) \quad \forall t \geq 0,$$

for some constant $K_* > 0$.

The complementary function (or conjugate function) \tilde{A} associated with A is given by the Legendre's transformation, that is,

$$\tilde{A}(s) = \max_{t \geq 0} \{st - A(t)\} \quad \text{for } s \geq 0.$$

The functions A and \tilde{A} are complementary each other. Moreover, we also have a Young type inequality given by

$$st \leq A(t) + \tilde{A}(s) \quad \forall s, t \geq 0. \quad (3.1)$$

In what follows, fixed an open set $\Omega \subset \mathbb{R}^N$ and a N-function A , we define the Orlicz space associated with A as

$$L^A(\Omega) = \left\{ u \in L_{loc}^1(\Omega) : \int_{\Omega} A\left(\frac{|u|}{\lambda}\right) dx < +\infty \text{ for some } \lambda > 0 \right\}.$$

The space $L^A(\Omega)$ is a Banach space endowed with Luxemburg norm given by

$$\|u\|_A = \inf \left\{ \lambda > 0 : \int_{\Omega} A\left(\frac{|u|}{\lambda}\right) dx \leq 1 \right\}.$$

The convexity of A implies in the inequality below, which will be used later on

$$\|u\|_A \leq 1 \iff \int_{\Omega} A(|u|) dx \leq 1. \quad (3.2)$$

Using the inequality (3.1), it is possible to prove a Hölder type inequality, that is,

$$\left| \int_{\Omega} uv dx \right| \leq 2\|u\|_A \|v\|_{\tilde{A}} \quad \forall u \in L^A(\Omega) \quad \text{and} \quad \forall v \in L^{\tilde{A}}(\Omega).$$

The space $L^A(\Omega)$ is separable and reflexive when A and \tilde{A} satisfy the Δ_2 -condition. Moreover, the Δ_2 -condition implies that

$$u_n \rightarrow u \text{ in } L^A(\Omega) \iff \int_{\Omega} A(|u_n - u|) dx \rightarrow 0.$$

3.1 The class $K_A(\Omega)$ and the subspace $E_A(\Omega)$

In the study of the Orlicz space $L^A(\Omega)$, we denote by $K_A(\Omega)$ the following set

$$K_A(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} : u \text{ is measurable and } \int_{\Omega} A(|u|) dx \leq 1 \right\}$$

and let us denote by $E_A(\Omega) \subset L^A(\Omega)$ the following subspace

$$E_A(\Omega) = \overline{L^{\infty}(\Omega)}^{\|\cdot\|_A} \quad \text{if } |\Omega| < +\infty \text{ is bounded}$$

or

$$E_A(\Omega) = \overline{C_0^{\infty}(\Omega)}^{\|\cdot\|_A} \quad \text{if } |\Omega| = +\infty \text{ is unbounded.}$$

Using the above notations, it follows that

$$E_A(\Omega) \subset K_A(\Omega) \subset L^A(\Omega),$$

and

$$K_A(\Omega) \subset \{u \in L^A(\Omega) : \text{dist}(u, E_A(\Omega)) \leq 1\}. \quad (3.3)$$

It is possible to prove that if A verifies Δ_2 -condition, then

$$E_A(\Omega) = K_A(\Omega) = L^A(\Omega).$$

However, if A does not satisfies the Δ_2 -condition, we have that $E_A(\Omega)$ is a proper subspace of $L^A(\Omega)$. For example, this situation holds for the N-function

$\Phi(t) = e^{|t|^2} - 1$, because it does not verify the Δ_2 -condition. Moreover, $L^\Phi(\Omega)$ is not reflexive, hence we cannot guarantee that if $J_0 \in (L^\Phi(\Omega))^*$, then

$$J_0(u) = \int_{\mathbb{R}^2} vu \, dx, \forall u \in L^\Phi(\mathbb{R}^2),$$

for some measurable function $v : \mathbb{R}^2 \rightarrow \mathbb{R}$. However, this type of problem does not hold in $(E_\Phi(\Omega))^*$, because if $J_1 \in (E_\Phi(\Omega))^*$ we know that there exists $v \in L^{\tilde{\Phi}}(\mathbb{R}^2)$ such that

$$J_1(u) = \int_{\mathbb{R}^2} vu \, dx, \forall u \in L^\Phi(\mathbb{R}^2).$$

Lemma 3.1 *Let $\xi(t) = \max\{t, t^2\}$ and $\tilde{\Phi}$ the conjugate function associated with Φ . Then,*

$$\tilde{\Phi}\left(\frac{\Phi(r)}{r}\right) \leq \Phi(r) \quad \text{and} \quad \tilde{\Phi}(tr) \leq \xi(t)\tilde{\Phi}(r), \quad t, r \geq 0.$$

Hence, $\tilde{\Phi} \in \Delta_2$, $E_{\tilde{\Phi}}(\mathbb{R}^2) = L_{\tilde{\Phi}}(\mathbb{R}^2)$ and $L_{\tilde{\Phi}}(\mathbb{R}^2)$ is separable.

Proof. The first inequality follows from [32]. To prove the second one, we recall that

$$2 \leq \frac{\Phi'(t)t}{\Phi(t)}, \quad t \in (0, +\infty).$$

Fix $s > 0$, such that $t = \tilde{\Phi}'(s)$. Since $\tilde{\Phi}' = (\Phi')^{-1}$ and $s\tilde{\Phi}'(s) = \tilde{\Phi}(s) + \Phi(\tilde{\Phi}'(s))$, we derive that

$$2 \leq \frac{\Phi'(\tilde{\Phi}'(s))\tilde{\Phi}'(s)}{\Phi(\tilde{\Phi}'(s))} = \frac{s\tilde{\Phi}'(s)}{s\tilde{\Phi}'(s) - \tilde{\Phi}(s)},$$

and so,

$$2s\tilde{\Phi}'(s) - 2\tilde{\Phi}(s) \leq s\tilde{\Phi}'(s),$$

that is

$$\frac{s\tilde{\Phi}'(s)}{\tilde{\Phi}(s)} \leq 2.$$

Now, fixing $s = \rho r > 0$, we get

$$\frac{d}{d\sigma} \left(\ln \left(\tilde{\Phi}(\rho r) \right) \right) \leq \frac{2}{\sigma}.$$

From this,

$$\tilde{\Phi}(tr) \leq t^2 \tilde{\Phi}(r), \quad t \geq 1 \text{ and } r \geq 0. \quad (3.4)$$

On the other hand, the convexity of $\tilde{\Phi}$ combines with $\tilde{\Phi}(0) = 0$ to give

$$1 \leq \frac{\tilde{\Phi}'(t)t}{\tilde{\Phi}(t)}, \quad t \in (0, +\infty).$$

Using again [32], we see that

$$\tilde{\Phi}(tr) \leq t\tilde{\Phi}(r), \quad t \in (0, 1] \text{ and } r \geq 0. \quad (3.5)$$

Hence, from (3.4) and (3.5),

$$\tilde{\Phi}(tr) \leq \xi(t)\tilde{\Phi}(r), \quad t, r \geq 0.$$

Now, the conclusion follows from [32]. \blacksquare

Lemma 3.2 *Let $X = H_0^1(\Omega)$ or $X = H^1(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain or $\Omega = \mathbb{R}^2$. Then, the embedding $X \hookrightarrow E_\Phi(\Omega)$ is continuous.*

Proof. From (2.2), (2.4) and (2.6), we have that the embedding $X \hookrightarrow L_\Phi(\Omega)$ is continuous. Then, the lemma follows by demonstrating the inclusion $X \subset E_\Phi(\Omega)$. First of all, by (2.1), (2.3) and (2.5), we know that

$$\int_{\Omega} (e^{\lambda|u|^2} - 1) dx < +\infty, \quad \forall \lambda \geq 0 \quad \text{and} \quad \forall u \in X,$$

implying that

$$X \subset K_\Phi(\Omega). \quad (3.6)$$

Assume by contradiction that there is $u_0 \in X$ with $u_0 \notin E_\Phi(\Omega)$. Since $E_\Phi(\Omega)$ is a closed subspace in $L^\Phi(\Omega)$, we ensure that $\text{dist}(u_0, E_\Phi(\Omega)) > 0$. Thereby, for

$$\lambda > \frac{1}{\text{dist}(u_0, E_\Phi(\Omega))},$$

we have that

$$\begin{aligned} \text{dist}(\lambda u_0, E_\Phi(\Omega)) &= \inf \{ \|\lambda u_0 - v\|_\Phi; v \in E_\Phi(\Omega) \} \\ &= \lambda \inf \left\{ \|u_0 - \frac{v}{\lambda}\|_\Phi; \frac{v}{\lambda} \in E_\Phi(\Omega) \right\} \\ &= \lambda \text{dist}(u_0, E_\Phi(\Omega)) > 1. \end{aligned}$$

Then, by (3.3), $\lambda u_0 \notin K_\Phi(\Omega)$, which contradicts (3.6), because $\lambda u_0 \in X$. \blacksquare

Lemma 3.3 *The embeddings $E_\Phi(\Omega) \hookrightarrow L^{2n}(\Omega)$ are continuous for any $n \in \mathbb{N}$.*

Proof. For each $n \in \mathbb{N}^*$, we know that

$$\frac{1}{n!} t^{2n} \leq \sum_{k=0}^{+\infty} \frac{1}{k!} t^{2k} = e^{t^2} - 1, \quad \forall t \in \mathbb{R}.$$

Then, for each $u \in E_\Phi(\Omega)$

$$\frac{1}{n!} \int_{\Omega} \left(\frac{u}{|u|_\Phi} \right)^{2n} dx \leq \int_{\Omega} \left(e^{\left(\frac{u}{|u|_\Phi} \right)^2} - 1 \right) dx \leq 1,$$

leading to

$$|u|_{2n}^{2n} = \int_{\Omega} |u|^{2n} \leq n! |u|_\Phi^{2n},$$

showing the lemma. \blacksquare

4 Some properties of the functional Ψ

Let $f : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$ be a measurable function for each $t \in \mathbb{R}$ and Locally Lipschitzian for each $x \in \mathbb{R}^2$ verifying:

(f_2) There is $t_0 \geq 0$ such that

$$f(x, t) = 0 \quad \text{for} \quad t < t_0 \quad \text{and} \quad \forall x \in \mathbb{R}^2$$

and

$$f(x, t) > 0 \quad \text{for} \quad t > t_0 \quad \text{and} \quad \forall x \in \mathbb{R}^2.$$

(f_*) There are $\alpha_0, c_1, c_2 > 0$ and $\alpha_0 > 0$ such that

$$|\xi| \leq c_1|u| + c_2e^{\alpha_0|u|^2}, \quad \forall \xi \in \partial_t F(x, u) \quad \text{and} \quad \forall x \in \mathbb{R}^2,$$

where $F(x, t) = \int_0^t f(x, s)ds$.

Theorem 4.1 *The functional $\Psi : E_\Phi(\Omega) \rightarrow \mathbb{R}$ given by*

$$\Psi(u) = \int_\Omega F(x, u)dx$$

is well defined and $\Psi \in Lip_{loc}(E_\Phi(\Omega), \mathbb{R})$.

Proof. For each $u \in E_\Phi(\Omega)$ and $R > 0$, consider $w, v \in B_R(u) \subset E_\Phi(\Omega)$. By Lebourg's Theorem, there is $\xi \in \partial F_t(x, \theta)$ with $\theta \in [w, v]$ such that

$$|F(x, w) - F(x, v)| = |\langle \xi, w - v \rangle| \leq |\xi||w - v|.$$

Then by (f_*),

$$|F(x, w) - F(x, v)| \leq (c_1|\theta| + c_2(e^{\alpha|\theta|^2} - 1))|w - v|,$$

for $\alpha > \alpha_0$ and α close to α_0 . Setting $\eta(x) = |v(x)| + |w(x)|$, it follows that

$$|F(x, w) - F(x, v)| \leq (c_1|\eta| + c_2(e^{\alpha|\eta|^2} - 1))|w - v|,$$

and so,

$$|\Psi(w) - \Psi(v)| \leq \int_\Omega (c_1|\eta| + c_2(e^{\alpha|\eta|^2} - 1))|w - v|dx.$$

By Hölder's inequality,

$$|\Psi(w) - \Psi(v)| \leq c_1|w - v|_2 + c_2 \left(\int_\Omega (e^{2\alpha|\eta|^2} - 1)dx \right)^{\frac{1}{2}} |w - v|_2.$$

Once $E_\Phi(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, see Lemma 3.3, we derive that

$$\begin{aligned} |\Psi(w) - \Psi(v)| &\leq c_1(|w - u|_\Phi + |u - v|_\Phi + 2|u|_\Phi)|w - v|_\Phi \\ &\quad + c_2 \left(\int_\Omega (e^{2\alpha|\eta|^2} - 1)dx \right)^{\frac{1}{2}} |w - v|_\Phi. \end{aligned} \quad (4.1)$$

On the other hand, the convexity of Φ yields

$$\begin{aligned} \int_{\Omega} \left(e^{2\alpha(|w|+|v|)^2} - 1 \right) dx &\leq \frac{1}{4} \int_{\Omega} \left(e^{32\alpha|w-u|^2} - 1 \right) dx + \frac{1}{4} \int_{\Omega} \left(e^{32\alpha|v-u|^2} - 1 \right) dx \\ &\quad + \frac{1}{2} \int_{\Omega} \left(e^{32\alpha|u|^2} - 1 \right) dx. \end{aligned} \quad (4.2)$$

Now, fixing $R > 0$ verifying $R < \frac{1}{\sqrt{32\alpha_0}}$ and α close to α_0 satisfying $R < \frac{1}{\sqrt{32\alpha}}$, we derive that

$$|\sqrt{32\alpha}(w-u)|_{\Phi} \leq \sqrt{32\alpha}R \leq 1 \text{ and } |\sqrt{32\alpha}(v-u)|_{\Phi} \leq \sqrt{32\alpha}R \leq 1,$$

and so,

$$\int_{\Omega} \left(e^{32\alpha|w-u|^2} - 1 \right) dx \leq 1 \quad \text{and} \quad \int_{\Omega} \left(e^{32\alpha|v-u|^2} - 1 \right) dx \leq 1, \quad (4.3)$$

for all $w, v \in B_R(u)$. From (4.2)-(4.3)

$$\int_{\Omega} \left(e^{2\alpha(|w|+|v|)^2} - 1 \right) dx \leq \frac{1}{4} \left(2 + 2 \int_{\Omega} \left(e^{32\alpha|u|^2} - 1 \right) dx \right). \quad (4.4)$$

Thereby, gathering (4.1) and (4.4),

$$\begin{aligned} |\Psi(w) - \Psi(v)| &\leq c_1 (R + R + 2|u|_{\Phi}) |w - v|_{\Phi} \\ &\quad + c_2 \frac{1}{4} \left(2 + 2 \int_{\Omega} \left(e^{32\alpha|u|^2} - 1 \right) dx \right)^{\frac{1}{2}} |w - v|_{\Phi} \\ &= K(R, u) |w - v|_{\Phi}, \quad \forall w, v \in B_R(u). \end{aligned}$$

■

Our next goal is proving the differentiable inclusion

$$\partial\Psi(u) \subset \int_{\Omega} \partial_t F(x, u) dx, \quad u \in E_{\Phi}(\Omega). \quad (4.5)$$

To do this, we need of the following result

Lemma 4.1 *Let $\psi : \mathbb{R} \rightarrow \mathbb{R}_+$ be a N -function and*

$$g_n \rightarrow g \text{ in } E_{\psi}(\Omega).$$

Then, there is $\hat{g} \in E_{\psi}(\Omega)$ and a subsequence of $\{g_n\}$, denoted by $\{g_{m_k}\}$, such that

- (i) $g_{m_k}(x) \rightarrow g(x) \quad a.e. \quad x \in \Omega,$
- (ii) $|g_{m_k}(x)| \leq \hat{g}(x) \quad a.e. \quad x \in \Omega.$

Proof. As

$$|g_m - g|_\psi \rightarrow 0,$$

we have that

$$\int_{\Omega} \psi(g_m - g) dx \leq |g_m - g|_\psi \rightarrow 0,$$

implying that there is a subsequence of $\{g_n\}$, denoted by $\{g_{m_k}\}$, such that

$$\psi(g_{m_k} - g)(x) \rightarrow 0 \quad \text{a.e. in } \Omega,$$

and so,

$$(g_{m_k} - g)(x) = \psi^{-1} \circ \psi(|g_{m_k} - g|)(x) \rightarrow 0 \quad \text{a.e. in } \Omega,$$

that is,

$$g_{m_k}(x) \rightarrow g(x) \quad \text{a.e. in } \Omega,$$

Now, define

$$\zeta_m = \sum_{k=1}^m |g_{n_{k+1}} - g_{n_k}| \in E_\psi(\Omega),$$

with

$$|g_{n_{k+1}} - g_{n_k}|_\psi < \frac{1}{2^k}, \quad \forall k \in \mathbb{N}.$$

Hereafter, g_k denotes g_{n_k} , that is, $g_k := g_{n_k}$. For $n \leq m$,

$$|\zeta_m - \zeta_n|_\psi \leq \sum_{k=n}^m |g_{k+1} - g_k|_\psi \leq \sum_{k=n}^m \frac{1}{2^k} \rightarrow 0,$$

from it follows that $\{\zeta_m\} \subset E_\psi(\Omega)$ is a Cauchy's sequence in $E_\psi(\Omega)$. Once $E_\psi(\Omega)$ is a Banach space, there exists $\zeta \in E_\psi(\Omega)$ such that

$$\zeta_m \rightarrow \zeta \text{ in } E_\psi(\Omega).$$

Then

$$\int_{\Omega} \psi(\xi_m - \xi) dx \leq |\zeta_m - \zeta|_\psi \rightarrow 0,$$

and so,

$$\zeta_{m_k}(x) \rightarrow \zeta(x) \text{ a.e. in } \Omega,$$

and

$$\zeta_{m_k}(x) \leq \zeta(x) \text{ a.e. in } \Omega, \quad k \in \mathbb{N}.$$

On the other hand, for $n \leq m$,

$$|g_m - g_n|(x) \leq \xi_m(x) \leq \zeta(x) \text{ a.e. in } \Omega.$$

Setting $\hat{g} = \zeta + |g| \in E_\psi(\Omega)$ and taking the $n \rightarrow +\infty$, we get

$$|g_m(x)| \leq \hat{g}(x) \text{ a.e. in } \Omega \quad \forall m \in \mathbb{N},$$

showing (ii). ■

Theorem 4.2 Assume (f_*) and that $\underline{f}(x, t)$ and $\overline{f}(x, t)$ are N - measurable functions. If $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain or $\Omega = \mathbb{R}^2$, then for each $u \in E_\Phi(\Omega)$,

$$\partial\Psi(u) \subset \partial_t F(x, u) = [\underline{f}(x, u(x)), \overline{f}(x, u(x))] \text{ a.e. in } \Omega. \quad (4.6)$$

Moreover,

$$\partial\Psi|_X(u) \subset \partial\Psi(u), \quad u \in X,$$

where $X = H_0^1(\Omega)$ or $X = H^1(\Omega)$. Here, the above inclusion means that given $\xi \in \partial\Psi(u) \subset E_\Phi(\Omega)^*$, there is $\tilde{\xi} \in L^{\tilde{\Phi}}(\Omega)$ satisfying

- $\langle \xi, v \rangle = \int_\Omega \tilde{\xi} v dx, \quad \forall v \in E_\Phi(\Omega),$
- $\tilde{\xi}(x) \in \partial_t F(x, u) = [\underline{f}(x, u(x)), \overline{f}(x, u(x))] \text{ a.e. in } \Omega.$

Proof. Given $u, v \in E_\Phi(\Omega)$, let $\{g_j\} \subset E_\Phi(\Omega)$ with $g_j \rightarrow 0$ in $E_\Phi(\Omega)$ and $\{\lambda_j\} \subset \mathbb{R}_+$ with $\lambda_j \rightarrow 0$ verifying

$$\Psi^0(u; v) = \lim_{j \rightarrow +\infty} \int_\Omega \frac{F(u + g_j + \lambda_j v) - F(u + g_j)}{\lambda_j} dx. \quad (4.7)$$

Setting

$$F_j(u; v) := \frac{F(u + g_j + \lambda_j v) - F(u + g_j)}{\lambda_j},$$

the Lebourg's Theorem guarantees that there is $\xi_j \in \partial_t F(x, \theta_j)$, with $\theta_j \in [u + g_j + \lambda_j v, u + g_j]$ such that

$$|F_j(u; v)| = \frac{1}{\lambda_j} |\langle \xi_j, \lambda_j v \rangle| \leq |\xi_j| |v|.$$

Hence by (f_*) ,

$$|F_j(u; v)| \leq \left(c_1 |\theta_j| + c_2 (e^{\alpha |\theta_j|^2} - 1) \right) |v|,$$

for $\alpha > \alpha_0$ and α close to α_0 . Fixing

$$\beta_j = (|u| + |g_j| + \lambda_j |v|) + (|u| + |g_j|) = 2|u| + 2|g_j| + \lambda_j |v|,$$

we see that

$$|F_j(u; v)| \leq \left(c_1 |\beta_j| + c_2 (e^{\alpha |\beta_j|^2} - 1) \right) |v|. \quad (4.8)$$

Applying Lemma 4.1, there exists $g_* \in E_\Phi(\Omega)$ such that

$$|\beta_j| \leq 2|u| + 2g_* + c|v| \text{ a.e. in } \Omega, \quad (4.9)$$

for some subsequence. Thereby, by (4.8) and (4.9), there exists a subsequence $\{F_{j_k}(u; v)\}$ such that

$$|F_{j_k}(u; v)| \leq \left(c_1 (2|u| + 2g_* + c|v|) + c_2 (e^{\alpha (2|u| + 2g_* + c|v|)^2} - 1) \right) |v| \in L^1(\Omega).$$

Applying the Lebesgue's Theorem,

$$\begin{aligned}
\Psi^0(u; v) &= \lim_{j_k \rightarrow +\infty} \int_{\Omega} F_{j_k}(u; v) dx = \int_{\Omega} \lim_{j_k \rightarrow +\infty} F_{j_k}(u; v) dx \\
&\leq \int_{\Omega} F^0(u; v) dx = \int_{\Omega} \max\{\langle \xi, v \rangle; \xi \in \partial_t F(x, u)\} dx \\
&\leq \int_{[v < 0]} \underline{f}(x, u) v dx + \int_{[v > 0]} \overline{f}(x, u) v dx. \tag{4.10}
\end{aligned}$$

Now, we will show that for each $\xi \in \partial\Psi(u) \subset (E_{\Phi}(\Omega))^*$, the function $\tilde{\xi} \in L^{\tilde{\Phi}}(\Omega)$, which satisfies

$$\langle \xi, w \rangle = \int_{\Omega} \tilde{\xi} w dx, \quad \forall w \in E_{\Phi}(\Omega),$$

must verify

$$\tilde{\xi}(x) \in [\underline{f}(x, u(x)), \overline{f}(x, u(x))] \quad \text{a.e. in } \Omega.$$

Indeed, assume by contradiction that there is a measurable set $\mathcal{M} \subset \Omega$, with $0 < |\mathcal{M}| < +\infty$, satisfying

$$\tilde{\xi}(x) < \underline{f}(x, u(x)), \quad x \in \mathcal{M}. \tag{4.11}$$

Setting $v = -\chi_{\mathcal{M}} \in E_{\Phi}(\Omega)$, we must have

$$-\int_{\mathcal{M}} \tilde{\xi} dx = \int_{\Omega} \tilde{\xi} (-\chi_{\mathcal{M}}) dx \leq \Psi^0(u, -\chi_{\mathcal{M}}) \leq -\int_{\mathcal{M}} \underline{f}(x, u(x)) dx,$$

leading to

$$\int_{\Omega} \tilde{\xi} \chi_{\mathcal{M}} dx \geq \int_{\mathcal{M}} \underline{f}(x, u(x)) dx,$$

which contradicts (4.11). Thereby,

$$\tilde{\xi}(x) \geq \underline{f}(x, u(x)) \quad \text{a.e. in } \Omega.$$

The same type of arguments work to show that

$$\tilde{\xi}(x) \leq \overline{f}(x, u(x)) \quad \text{a.e. in } \Omega.$$

From definition of X , we know that $\overline{X}^{\|\cdot\|_{\Phi}} = E_{\Phi}(\Omega)$, then the Lemma 3.2 combined with chain rule gives

$$\partial\Psi|_X(u) \subset \partial\Psi(u), \quad \forall u \in X.$$

■

5 An application

In this section, we will study the existence of solution for the following class of multivalued elliptic equation

$$-\Delta u + V(x)u - \epsilon h(x) \in \partial_t F(x, u), \quad \text{in } \mathbb{R}^2, \quad (P)$$

where

- $h \in H^{-1}$, that is, the functional $\langle h, v \rangle = \int_{\mathbb{R}^2} h v dx$ is continuous in $H^1(\mathbb{R}^2)$ and $0 < \int_{\mathbb{R}^2} h dx < +\infty$.
- $F(x, t) = \int_0^t f(x, s) ds$, for $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$, where f verifies (f_0) and (f_1) .

Related to the potential $V : \mathbb{R}^2 \rightarrow \mathbb{R}$, we assume that

(V₁) V is continuous and $V(x) \geq V_0 > 0$, $\forall x \in \mathbb{R}^2$,

(V₂) $\frac{1}{V} \in L^1(\mathbb{R}^2)$.

In order to apply variational methods, we will consider the Hilbert space

$$E := \left\{ u \in H^1(\mathbb{R}^2) / \int_{\mathbb{R}^2} V(x)|u|^2 dx < +\infty \right\}$$

endowed with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^2} (\nabla u \nabla v + V(x)uv) dx.$$

Associated with the above inner product, we have the norm

$$\| u \| = \left(\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2) dx \right)^{\frac{1}{2}}.$$

using the above information, it is well known that

(E₁) $E \hookrightarrow L^q(\mathbb{R}^2)$ is a compact embedding for all $q \geq 1$, see [28, 29]

(E₂) $E \hookrightarrow H^1(\mathbb{R}^2) \hookrightarrow E_{\Phi}(\Omega)$ is a continuous embedding (see Lemma 3.2).

In the present paper, we say that $u \in E$ is a solution for (P), if there is $\rho \in L_{\Phi}(\mathbb{R}^2)$ such that

- (i) $\int_{\mathbb{R}^2} (\nabla u \nabla v + V(x)uv) dx - \int_{\mathbb{R}^2} \rho v dx - \epsilon \int_{\mathbb{R}^2} h v dx = 0$, $v \in E$,
- (ii) $\rho(x) \in \partial_t F(x, u(x))$ a.e. in \mathbb{R}^2 ,

(iii) $||[u > t_0]|| > 0$.

The reader is invited to observe that $u \in E$ is a solution for (P) if, and only if, u is a critical point of the energy functional associated with (P) given by:

$$I_\epsilon(u) = \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2)dx - \int_{\mathbb{R}^2} F(x, u)dx - \epsilon \int_{\mathbb{R}^2} h v dx, \quad u \in E.$$

Note that Theorem 4.1 gives that $I_\epsilon \in Lip_{loc}(E; \mathbb{R})$. From this, using some properties of the generalizity gradient together with Theorem 4.2, given $u \in E$ and $w \in \partial I_\epsilon(u)$, there exists $\rho \in L_{\tilde{\Phi}}(\mathbb{R}^2)$ such that

$$\langle w, v \rangle = \int_{\mathbb{R}^2} (\nabla u \nabla v + V(x)uv)dx - \int_{\mathbb{R}^2} \rho v dx - \epsilon \int_{\mathbb{R}^2} h v dx \quad \forall v \in E,$$

with

$$\rho(x) \in \partial_t F(x, u(x)) \quad \text{a.e. in } \mathbb{R}^2.$$

6 Existence of solution via Ekeland's variational principle

In this section, we will get a solution via Ekeland's variational principle.

Lemma 6.1 *Assume that (f_0) and $(f_2) - (f_4)$ hold. Then, there are $\epsilon_0, r, \alpha, \delta > 0$, such that*

$$c_\epsilon = \inf_{\|u\| \leq r} I_\epsilon(u) < -\delta$$

and

$$I_\epsilon(u) \geq \alpha \quad \text{for } \|u\| = r$$

for all $\epsilon \in (0, \epsilon_0)$. Here, r is independent of ϵ , but α and δ depend on ϵ . Moreover, the numbers ϵ_0, r, α and δ do not depend on t_0 given in (f_2) .

Proof. Using the conditions on F , given $\beta \in (0, \lambda_1)$, $q > 2$ and $\alpha > \alpha_0$ close to α_0 , we have that

$$F(x, t) \leq \frac{(\lambda_1 - \beta)}{2} |t|^2 + C |t|^q (e^{\alpha |t|^2} - 1), \quad \forall t \in \mathbb{R}.$$

Then, fixing $r > 0$ small enough such that $\alpha r^2 < 4\pi$ and using Lemma 2.2, we get for $u \in E$ with $\|u\| \leq r$,

$$\begin{aligned} I_\epsilon(u) &\geq \frac{1}{2} \|u\|^2 - \frac{(\lambda_1 - \beta)}{2} |u|_2^2 - C \|u\|^q - \epsilon \|h\|_* \|u\| \\ &= \frac{1}{2} \left(1 - \frac{(\lambda_1 - \beta)}{\lambda_1} \right) \|u\|^2 - C \|u\|^q - \epsilon \|h\|_* \|u\|, \end{aligned}$$

showing that I_ϵ is bounded from below for $\|u\| \leq r$. Moreover, decreasing r if necessary of a way that

$$\frac{1}{2}r^2 - Cr^q \geq \frac{1}{4}r^2,$$

we derive that

$$I_\epsilon(u) \geq \frac{1}{4}r^2 - \epsilon\|h\|r, \quad \|u\| = r.$$

Thereby, choosing $\epsilon_0 > 0$ such that

$$\alpha_\epsilon = \frac{1}{4}r^2 - \epsilon\|h\|r > 0, \quad \forall \epsilon \in (0, \epsilon_0),$$

we see that

$$I_\epsilon(u) \geq \alpha_\epsilon \quad \text{for } \|u\| = r, \quad \forall \epsilon \in (0, \epsilon_0).$$

Now, take $v \in E$ satisfying

$$\|v\| = 1 \quad \text{and} \quad \int_{\mathbb{R}^2} hv \, dx > 0.$$

Note that for each $s > 0$,

$$I_\epsilon(sv) = \frac{s^2}{2} - \int_{\mathbb{R}^2} F(x, sv) \, dx - \epsilon s \int_{\mathbb{R}^2} hv \, dx < \frac{s^2}{2} - \epsilon s \int_{\mathbb{R}^2} hv \, dx.$$

Fixing $s = s(\epsilon) > 0$ small enough satisfying

$$\delta = -\frac{s^2}{2} + \epsilon s \int_{\mathbb{R}^2} hv \, dx > 0,$$

it follows that $\|sv\| < r$ and

$$I_\epsilon(sv) < -\delta < 0,$$

implying that

$$c_\epsilon = \inf_{\|u\| \leq r} I_\epsilon(u) < -\delta < 0.$$

■

Theorem 6.1 *Assume $(V_1), (V_2), (f_0), (f_2)$ and (f_3) . Then, problem (P) possesses a solution $u_\epsilon \in E$, with $I_\epsilon(u_\epsilon) = c_\epsilon < -\delta < 0$, for all $\epsilon \in (0, \epsilon_0)$ and $t_0 \in [0, t_*)$, with $t_* = t_*(\epsilon) = \frac{2\delta}{\epsilon \int_{\mathbb{R}^2} hv \, dx} > 0$.*

Proof. Fix $r > 0$ such that $\alpha_0 r^2 < 4\pi$. Applying the Lemma 6.1 together with Ekeland's variational principle, there is $\{u_n\} \subset \overline{B}_r(0)$ verifying

- $I_\epsilon(u_n) \rightarrow c_\epsilon$ (as $n \rightarrow +\infty$),
- $\lambda_\epsilon(u_n) := \min\{\|\xi\|_{E^*} / \xi \in \partial I_\epsilon(u_n)\} \rightarrow 0$ (as $n \rightarrow +\infty$).

Next, we fix $w_n \in \partial I_\epsilon(u_n)$ and $\{\rho_n\} \subset L_{\tilde{\Phi}}(\mathbb{R}^2)$ verifying

$$\|w_n\|_{E^*} := \lambda_\epsilon(u_n)$$

$$\langle w_n, v \rangle = \int_{\mathbb{R}^2} \nabla u_n \nabla v + V(x) u_n v dx - \int_{\mathbb{R}^2} \rho_n v dx - \epsilon \int_{\mathbb{R}^2} h v dx, \quad \forall v \in E, \quad (6.1)$$

and

$$\rho_n(x) \in \partial_t F(x, u_n(x)) \quad \text{a.e. in } \mathbb{R}^2.$$

We claim that $\{\rho_n\}$ is bounded in $L_{\tilde{\Phi}}(\mathbb{R}^2)$. Indeed, fixing $p > 4$, $\alpha > \alpha_0$ with $\alpha r^2 < 4\pi$, and using (f_*) , we get

$$\int_{\mathbb{R}^2} \tilde{\Phi}(\rho_n) dx \leq \int_{\mathbb{R}^2} \tilde{\Phi} \left(c_1 |u_n| + c_2 |u_n|^p (e^{\alpha |u_n|^2} - 1) \right) dx.$$

The convexity of $\tilde{\Phi}$ and the Δ_2 -condition combine to give

$$\int_{\mathbb{R}^2} \tilde{\Phi}(\rho_n) dx \leq \frac{\xi(2c_1)}{2} \int_{\mathbb{R}^2} \tilde{\Phi}(|u_n|) dx + \frac{\xi(2c_2)}{2} \int_{\mathbb{R}^2} \tilde{\Phi} \left(|u_n|^p (e^{\alpha |u_n|^2} - 1) \right) dx.$$

By Lemma 3.1 and (E_1) , there are positive constants C_1, C_2 such that

$$\begin{aligned} \int_{\mathbb{R}^2} \tilde{\Phi}(\rho_n) dx &\leq C_1 (|u_n|_1 + |u_n|_2^2) + \\ &+ C_2 \int_{\mathbb{R}^2} (|u_n|^{p+1} + |u_n|^{2(p+1)}) (e^{\alpha_0 |u_n|^2} - 1) dx. \end{aligned}$$

Recalling that the space E is continuously embedding in $L^1(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$, and $\alpha r^2 < 4\pi$, the Lemma 2.2 yields there is $C_3 > 0$ verifying

$$\int_{\mathbb{R}^2} \tilde{\Phi}(\rho_n) dx \leq C_3, \quad \forall n \in \mathbb{N},$$

showing that $\{\rho_n\}$ is a bounded sequence in $L_{\tilde{\Phi}}(\mathbb{R}^2)$. From this, the sequence of functionals $\{\tilde{\rho}_n\} \subset \partial \Psi(u_n) \subset (E_{\Phi}(\mathbb{R}^2))^*$ associated with $\{\rho_n\}$ is also bounded in $(E_{\Phi}(\mathbb{R}^2))^*$, and so, there is $\tilde{\rho}_0 \in (E_{\Phi}(\mathbb{R}^2))^*$, such that $\tilde{\rho}_n \xrightarrow{*} \tilde{\rho}_0$ in $(E_{\Phi}(\mathbb{R}^2))^*$ for some subsequence, that is,

$$\int_{\mathbb{R}^2} \rho_n v dx = \langle \tilde{\rho}_n, v \rangle \rightarrow \langle \tilde{\rho}_0, v \rangle = \int_{\mathbb{R}^2} \rho_0 v dx, \quad \forall v \in E, \quad (6.2)$$

for some $\rho_0 \in L_{\tilde{\Phi}}(\mathbb{R}^2)$.

Now, using the fact that $\{u_n\}$ is also bounded in E , there is $u_\epsilon \in E$ such that

$$u_n \rightharpoonup u_\epsilon \text{ in } E. \quad (6.3)$$

From (6.1)-(6.3)

$$0 = \int_{\mathbb{R}^2} \nabla u_\epsilon \nabla v + V(x) u_\epsilon v dx - \int_{\mathbb{R}^2} \rho_0 v dx - \epsilon \int_{\mathbb{R}^2} h v dx, \quad v \in E. \quad (6.4)$$

To conclude the proof that u_ϵ is a solution of (P) , we must prove that

i) $\rho_0(x) \in \partial_t F(x, u_\epsilon(x))$ a.e. in \mathbb{R}^2 and

ii) $||[u_\epsilon > t_0]|| > 0$.

To prove the i), we must show that $\{u_n\}$ is strongly convergent to u_ϵ in E , because this fact will imply that $\tilde{\rho}_0 \in \partial\Psi(u_0)$. This way, by Theorem 4.2

$$\rho_0(x) \in \partial_t F(x, u_\epsilon(x)) \text{ a.e in } \mathbb{R}^2.$$

Related to the second item, the proof is as follows: If $t_0 = 0$, then $||[u_\epsilon > t_0]|| > 0$, because $u_\epsilon \geq 0$ and $u_\epsilon \neq 0$. Next, we will consider the case $t_0 \in (0, t_*)$. Once $\rho_0, u_\epsilon \geq 0$, it follows that

$$\begin{aligned} 0 &= ||u_\epsilon||^2 - \int_{\mathbb{R}^2} \rho_0 u_\epsilon dx - \epsilon \int_{\mathbb{R}^2} h u_\epsilon dx \\ &\leq ||u_\epsilon||^2 - \epsilon \int_{\mathbb{R}^2} h u_\epsilon dx, \end{aligned}$$

that is,

$$||u_\epsilon||^2 \geq \epsilon \int_{\mathbb{R}^2} h u_\epsilon dx. \quad (6.5)$$

Arguing by contradiction, we assume that $||[u_\epsilon > t_0]|| = 0$, for some $t_0 \in (0, t_*)$. Thereby,

$$f(x, u_\epsilon(x)) = 0, \text{ a.e. in } \mathbb{R}^2,$$

from where it follows that

$$\partial_t F(x, u_\epsilon(x)) = \{0\} \text{ a.e. in } \mathbb{R}^2.$$

Consequently,

$$\rho_0(x) = 0 \text{ a.e. in } \mathbb{R}^2.$$

On the other hand, by Lemma 6.1 and (6.5),

$$0 > -\delta > I_\epsilon(u_\epsilon) = \frac{1}{2} ||u_\epsilon||^2 - \epsilon \int_{\mathbb{R}^2} h u_\epsilon dx \geq -\frac{1}{2} \epsilon \int_{\mathbb{R}^2} h u_\epsilon dx \geq -\frac{t_0}{2} \epsilon \int_{\mathbb{R}^2} h dx,$$

implying that

$$t_0 \geq \frac{2\delta}{\epsilon \int_{\mathbb{R}^2} h dx} = t_*,$$

which is a contradiction.

Convergence of $\{u_n\}$ to u_ϵ in E : Hereafter, fix $\gamma_n := u_n - u_0$ and recall that $\gamma_n \rightharpoonup 0$ in E . By a direct computation,

$$||u_n||^2 = ||u_\epsilon||^2 + ||\gamma_n||^2 + o_n(1).$$

Moreover, we also have

$$\begin{aligned}
o_n(1) = \langle w_n, u_n \rangle &= \|u_n\|^2 - \int_{\mathbb{R}^2} \rho_n u_n dx - \epsilon \int_{\mathbb{R}^2} h u_n dx \\
&\quad - \|u_\epsilon\|^2 + \int_{\mathbb{R}^2} \rho_0 u_\epsilon dx + \epsilon \int_{\mathbb{R}^2} h u_\epsilon dx \\
&= \|\gamma_n\|^2 + \left(\int_{\mathbb{R}^2} \rho_0 u_\epsilon dx - \int_{\mathbb{R}^2} \rho_n u_\epsilon dx \right) \\
&\quad + \left(\int_{\mathbb{R}^2} \rho_n u_\epsilon dx - \int_{\mathbb{R}^2} \rho_n u_n dx \right) + o_n(1) \\
&= \|\gamma_n\|^2 - \int_{\mathbb{R}^2} \rho_n (u_n - u_\epsilon) dx + o_n(1) \\
&= \|\gamma_n\|^2 - \int_{\mathbb{R}^2} \rho_n \gamma_n dx + o_n(1). \tag{6.6}
\end{aligned}$$

On the other hand, by (f_1) ,

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} \rho_n \gamma_n dx \right| &\leq c_1 \int_{\mathbb{R}^2} |u_n| |\gamma_n| dx + c_2 \int_{\mathbb{R}^2} |\gamma_n| \left(e^{\alpha |u_n|^2} - 1 \right) dx \\
&\leq c_1 |u_n|_2^2 |\gamma_n|_2^2 + c_2 \int_{\mathbb{R}^2} |\gamma_n| \left(e^{\alpha |\gamma_n|^2} - 1 \right) dx.
\end{aligned}$$

Once $\alpha r^2 < 4\pi$, there is $q > 1$ close to 1, such that

$$M = \sup_{n \in \mathbb{N}} \left(\int_{\mathbb{R}^2} \left(e^{\alpha |\gamma_n|^2} - 1 \right)^q dx \right)^{\frac{1}{q}} < +\infty.$$

Thus, by Lemma 2.2 and Hölder inequality

$$\left| \int_{\mathbb{R}^2} \rho_n \gamma_n dx \right| \leq c_1 |u_n|_2^2 |\gamma_n|_2^2 + C |\gamma_n|_{q'},$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Since the embeddings $E \hookrightarrow L^2(\mathbb{R}^2)$ and $E \hookrightarrow L^{q'}(\mathbb{R}^2)$ are compact, we can ensure that

$$\int_{\mathbb{R}^2} \rho_n \gamma_n dx \rightarrow 0. \tag{6.7}$$

From (6.6) and (6.7), $\gamma_n \rightarrow 0$ in E , or equivalently $u_n \rightarrow u_\epsilon$ in E , finishing the proof. ■

7 Existence of solution via Mountain Pass

In this section, we will assume more some conditions on function f , namely $(f_0), (f_2) - (f_6)$. By (f_3) , there are $\tilde{\epsilon}, \tilde{\delta} := \tilde{\delta}_\epsilon > 0$, satisfying

$$2 \max \{ |\xi|; \xi \in \partial_t F(x, t) \} < (\lambda_1 - \tilde{\epsilon})|t|, \text{ for } |t| \leq \tilde{\delta} \text{ and } x \in \mathbb{R}^2.$$

From Lebourg's Theorem, there are $\theta(t) \in [0, t]$, with $|t| \leq \tilde{\delta}$ and $\xi_0 \in \partial F_t(x, \theta)$ verifying

$$|F(x, t)| = |F(x, t) - F(x, 0)| = |\xi_0||t - 0| \leq (\lambda_1 - \tilde{\epsilon})|t|, \quad x \in \mathbb{R}^2. \quad (7.1)$$

Now, by (f_0) , given $q \geq 2$ and $\alpha > \alpha_0$, there is $C = C(q, \tilde{\delta}) > 0$ such that

$$|\xi| \leq C|t|^{(q-1)} \left(e^{\alpha|t|^2} - 1 \right), \quad \xi \in \partial_t F(x, t), \quad |t| \geq \tilde{\delta} \text{ and } x \in \mathbb{R}^2.$$

Applying again Lebourg's Theorem

$$|F(x, t)| \leq C|t|^q \left(e^{\alpha|t|^2} - 1 \right), \quad \forall t \in \mathbb{R} \text{ and } \forall x \in \mathbb{R}^2.$$

From this, for $u \in E$ with $u \neq 0$ and $\|u\| := \eta_0 < \sqrt{\frac{4\pi}{\alpha_0}}$, we see that

$$\int_{\mathbb{R}^2} |F(x, u)| dx \leq C(q, \tilde{\delta}, \alpha_0) \|u\|^q. \quad (7.2)$$

Here, we have fixed α close to α_0 of the a way that $\alpha\eta_0^2 < 4\pi$.

Lemma 7.1 *Assume that (f_0) and $(f_2) - (f_6)$ hold. Then, there exists $\varphi_0 \in B_r^c(0)$ such that*

$$I_\epsilon(\varphi_0) < \inf_{\|u\|=r} I_\epsilon(u), \quad \epsilon \in (0, \epsilon_0],$$

where r and ϵ_0 are given in Lemma 6.1.

Proof. Let $\psi_0 \in C_0^\infty(\mathbb{R}^2) \setminus \{0\}$, $\psi_0 > 0$, with $\text{supt}(\psi_0) \subset K$, where $K \subset \mathbb{R}^2$ is the compact set fixed in (f_4) . In this case, for any $\epsilon > 0$,

$$I_\epsilon(t\psi_0) \leq \frac{t^2}{2} \|\psi_0\|^2 - c_3 t^\nu \int_{\mathbb{R}^2} \psi_0^\nu + c_4 |K| - t\epsilon \int_{\mathbb{R}^2} h\psi_0 dx,$$

from where it follows that

$$\lim_{t \rightarrow +\infty} I_\epsilon(t\psi_0) = -\infty.$$

Thus, the lemma follows choosing $\varphi_0 := t\psi_0 \in B_r^c(0)$ with t large enough. \blacksquare

From Lemmas 6.1 and 7.1, we can use the Mountain Pass Theorem to get a sequence $\{v_n\} \subset E$ verifying

$$I_\epsilon(v_n) \rightarrow d_\epsilon \text{ in } \mathbb{R} \text{ and } \lambda_\epsilon(v_n) := \max\{\|\xi\|_* / \xi \in \partial I_\epsilon(v_n)\} \rightarrow 0, \quad (7.3)$$

where

$$d_\epsilon := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I_\epsilon(\gamma(t)) \quad (\text{mountain pass level})$$

and

$$\Gamma := \{\gamma \in C([0, 1]; E) / \gamma(0) = 0 \text{ and } \gamma(1) = \varphi_0\}.$$

In the sequel, we intend to show that I_ϵ verifies the $(PS)_{d_\epsilon}$ condition if the parameter μ given in (f_6) is large enough. To this end, we need of the following lemma

Lemma 7.2 *Let $\{v_n\}$ be the sequence obtained in (7.3). Then, $\{v_n\}$ is bounded in E and*

$$\limsup_{n \rightarrow \infty} \|v_n\| \leq \frac{\left(\frac{\tau-1}{\tau}\right)\epsilon + \sqrt{\epsilon^2 \left(\frac{\tau-1}{\tau}\right)^2 + 2d_\epsilon \left(\frac{\tau-2}{\tau}\right)}}{\left(\frac{\tau-2}{\tau}\right)},$$

where τ is given in (f_5) .

Proof. Let $w_n \in E^*$ and $\rho_n \in \partial\Psi(v_n)$ verifying

$$\|w_n\|_* = \lambda_\epsilon(v_n) \quad \text{and} \quad \langle w_n, v_n \rangle = \|v_n\|^2 - \int_{\mathbb{R}^2} \rho_n v_n dx - \epsilon \int_{\mathbb{R}^2} h v_n dx.$$

From (f_5) and Theorem 4.2,

$$\begin{aligned} d_\epsilon + o_n(1) + o_n(1) \|v_n\| &\geq I_\epsilon(v_n) - \frac{1}{\tau} \langle w_n, u_n \rangle \\ &= \left(\frac{1}{2} - \frac{1}{\tau}\right) \|v_n\|^2 + \int_{\mathbb{R}^2} \frac{1}{\tau} \rho_n v_n - F(x, v_n) dx \\ &\quad + \left(\frac{1}{\tau} - 1\right) \epsilon \int_{\mathbb{R}^2} h v_n dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\tau}\right) \|v_n\|^2 + \left(\frac{1}{\tau} - 1\right) \epsilon \|h\|_* \|v_n\|, \end{aligned} \tag{7.4}$$

which implies that $\{v_n\}$ is bounded in E . Moreover, as $\{v_n\}$ does not converge to $v = 0$ in E , we can assume that for some subsequence,

$$l := \lim_{n \rightarrow \infty} \|v_n\| > 0.$$

Consequently, by (7.4),

$$d_\epsilon + \left(\frac{\tau-1}{\tau}\right) \epsilon \|h\|_* l \geq \left(\frac{1}{2} - \frac{1}{\tau}\right) l^2,$$

that is

$$\left(\frac{1}{2} - \frac{1}{\tau}\right) l^2 - \left(\frac{\tau-1}{\tau}\right) \epsilon \|h\|_* l - d_\epsilon \leq 0.$$

As $l > 0$, we must have

$$l \leq \frac{\left(\frac{\tau-1}{\tau}\right) \epsilon + \sqrt{\epsilon^2 \left(\frac{\tau-1}{\tau}\right)^2 + 4d_\epsilon \left(\frac{\tau-2}{2\tau}\right)}}{2 \left(\frac{\tau-2}{2\tau}\right)},$$

which completes the proof. ■

Lemma 7.3 *Assume $(f_0) - (f_6)$. Then, there are $\epsilon_1, \mu^* > 0$ and $t_1 > 0$ such that*

$$\frac{\left(\frac{\tau-1}{\tau}\right) \epsilon + \sqrt{\epsilon^2 \left(\frac{\tau-1}{\tau}\right)^2 + 2d_\epsilon \left(\frac{\tau-2}{\tau}\right)}}{\left(\frac{\tau-2}{\tau}\right)} < \sqrt{\frac{4\pi}{\alpha_0}}.$$

for all $\epsilon \in (0, \epsilon_1)$, $\mu \geq \mu^*$ and $t_0 \in [0, t_1)$.

Proof. Consider the function ψ_0 used in the proof of Lemma 7.1. Then,

$$\sup_{t \in [0, t_0]} I_\epsilon(t\psi_0) \leq \frac{t_0^2}{2} \|\psi_0\|^2,$$

and so, there is $t_2 > 0$ such that

$$\sup_{t \in [0, t_0]} I_\epsilon(t\psi_0) \leq \epsilon^2$$

for $t_0 \in [0, t_2)$. On the other hand, by (f₆),

$$\sup_{t \geq t_0} I_\epsilon(t\psi) \leq \max_{t \geq 0} \left\{ \frac{t^2}{2} \|\psi_0\|^2 - \mu t^p \int_{\mathbb{R}^2} \psi_0^p dx \right\} + \mu c_2 t_0^p |supt(\psi_0)|,$$

that is,

$$\sup_{t \geq t_0} I_\epsilon(t\psi_0) \leq \left(\frac{1}{2p^{\frac{2}{p-2}}} - \frac{1}{p^{\frac{p}{p-2}}} \right) \frac{1}{\mu^{\frac{2}{p-2}}} \left(\frac{\|\psi_0\|}{|\psi_0|_p} \right)^{\frac{2p}{p-2}} + \mu c_2 t_0^p |supt(\psi_0)|.$$

Now, fix $\mu^* > 0$ such that

$$\left(\frac{1}{2p^{\frac{2}{p-2}}} - \frac{1}{p^{\frac{p}{p-2}}} \right) \frac{1}{\mu^{\frac{2}{p-2}}} \left(\frac{\|\psi_0\|}{|\psi_0|_p} \right)^{\frac{2p}{p-2}} \leq \epsilon^2, \quad \forall \mu \geq \mu^*$$

and $t_3 = t_3(\mu, \epsilon) > 0$ such that

$$\mu c_2 t_0^p |supt(\psi_0)| \leq \epsilon^2, \quad \forall t \in [0, t_3].$$

From this, for $t_1 = \min\{t_2, t_3\}$, we must have

$$\sup_{t \geq t_0} I_\epsilon(t\psi_0) \leq 2\epsilon^2,$$

and so,

$$d_\epsilon \leq \max_{t \geq 0} I_\epsilon(t\psi_0) \leq 2\epsilon^2.$$

Hence, there is $c_1 > 0$ independent of ϵ such that

$$\frac{\left(\frac{\tau-1}{\tau}\right)\epsilon + \sqrt{\epsilon^2 \left(\frac{\tau-1}{\tau}\right)^2 + 2d_\epsilon \left(\frac{\tau-2}{\tau}\right)}}{\left(\frac{\tau-2}{\tau}\right)} \leq c_1 \epsilon.$$

Then, there is $\epsilon_0 > 0$ such that

$$\frac{\left(\frac{\tau-1}{\tau}\right)\epsilon + \sqrt{\epsilon^2 \left(\frac{\tau-1}{\tau}\right)^2 + 2d_\epsilon \left(\frac{\tau-2}{\tau}\right)}}{\left(\frac{\tau-2}{\tau}\right)} < \sqrt{\frac{4\pi}{\alpha_0}}, \quad \forall \epsilon \in (0, \epsilon_0).$$

■

As an immediate consequence of the last lemma, we have the following corollary.

Corollary 7.1 *Let $\{v_n\}$ be the sequence obtained in (7.3). Then, there is ϵ_0 such that*

$$\limsup_{n \rightarrow +\infty} \|v_n\|^2 < \frac{4\pi}{\alpha_0}, \quad \forall \epsilon \in (0, \epsilon_0).$$

Moreover, there is a subsequence of $\{v_n\}$ still denoted by itself, and $v_\epsilon \in E$ such that $v_n \rightarrow v_\epsilon$ in E .

Proof. The first part of the lemma is an immediate consequence of Lemmas 7.2 and 7.3. The proof of the second part follows the same idea explored in the proof of Theorem 6.1. ■

Theorem 7.1 *Assume $(V_1) - (V_2)$ and $(f_0) - (f_6)$. Then, there are ϵ_0, μ^* and $t_1 > 0$, such that problem (P) possesses a solution $v_\epsilon \in E$, with $I_\epsilon(v_\epsilon) = d_\epsilon > 0$, for all $\epsilon \in (0, \epsilon_0)$, $t_0 \in [0, t_1]$ and $\mu \geq \mu^*$. Moreover, decreasing ϵ_0 and t_1 , and increasing μ^* , if necessary, we have two solutions $u_\epsilon, v_\epsilon \in E$ with*

$$I_\epsilon(u_\epsilon) = c_\epsilon < 0 < d_\epsilon = I_\epsilon(v_\epsilon).$$

Proof. The theorem follows applying the Lemmas 6.1 and 7.1 and Corollary 7.1. ■

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Claudianor O. Alves and Jefferson A. Santos
Universidade Federal de Campina Grande,
Unidade Acadêmica de Matemática ,
CEP:58109-970, Campina Grande - PB, Brazil
e-mail: coalves@mat.ufcg.edu.br and jefferson@mat.ufcg.edu.br